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Dynamical systems with SO(4) and SO(3, 1) symmetry

M M Enikova and V I Karloukovski
JINR, Dubna, Moscow, USSR

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Abstract. Dynamical systems with three degrees of freedom and a Hamiltonian quadratic in momenta and invariant under space rotations are considered. A class of such systems which can be interpreted as a non-linear realisation of dynamical SO(4) or SO(3, 1) symmetry groups with a 'Runge–Lenze vector' linear in momenta is found. The corresponding equations of motion are solved explicitly.

1. Introduction

Interest in classical mechanics has never ceased, and in the last few years it has even increased. Different reasons may be found to explain this line of development. We shall point out one; if not the most important, it is at least the reason for our interest in the particular mechanical problem we solve here. This is the increasing significance in physics of any kind of non-linear problem, and the solution of non-linear field theories in particular. Classical mechanics is the most developed and successful example of a non-linear classical theory. Its general principles and methods are clearly formulated. However, we are now in a position of knowing a lot more about interactions other than those on which the whole Newtonian world was based and of being faced with new phenomena which suggest the search and study of unconventional forms of interaction that would have remained outside the scope of mechanics in previous centuries. It would not sound eccentric at present to discuss velocity-dependent forces and position-dependent masses, for instance, as we have done in a recent work (Karloukovski 1978). There we found all the dynamical systems with strict Kepler symmetry, i.e. systems whose Hamiltonian is of the form

$$H = \tfrac{1}{2} g^{ij} p_i p_j + \mathcal{U}, \tag{1.1}$$

where

$$g^{ij}(\boldsymbol{x}) = G_1(x^2)\delta^{ij} + G_2(x^2)x^i x^j, \tag{1.2}$$

with 'hidden' SO(4) or SO(3, 1) symmetry and a Runge–Lenz vector quadratic in momenta.

The present work is devoted to the SO(4) or SO(3, 1) symmetric dynamical systems of the form (1.1), (1.2) with a 'Runge–Lenz vector' linear in momenta. After explaining some general relations and notation in § 2, we discuss in § 3 all dynamical systems of this type, and solve in § 4 the equations of motion. In § 5 we briefly discuss the behaviour of the trajectories and the motion under the symmetry transformations.

The subclass of the SO(4) invariant dynamical systems, which we call the general case, is the mechanical analogue of non-linear realisations of the chiral SU(2) × SU(2) symmetry in field theory (Weinberg 1968). This allows us to take the results obtained in this work back to field theory and to construct a large family of exact finite-energy solutions in classical chiral field theories (Velchev *et al* 1978).

2. General relations and notation

We study here classical dynamical systems with three degrees of freedom, $\boldsymbol{x} = (x^1, x^2, x^3) = (x_1, x_2, x_3)$, with the Lagrangian

$$L = \tfrac{1}{2} g_{ij}(\boldsymbol{x}) \dot{x}^i \dot{x}^j - \mathcal{U}(\boldsymbol{x}) \tag{2.1}$$

or the Hamiltonian

$$H = \tfrac{1}{2} g^{ij}(\boldsymbol{x}) p_i p_j + \mathcal{U}(\boldsymbol{x}). \tag{2.2}$$

The contravariant and covariant metric tensors are related by

$$g^{in} g_{nj} = \delta^i_j = g_{jn} g^{ni}. \tag{2.3}$$

We shall assume that the angular momentum

$$J_j = \epsilon^l_{jk} x^k p_l \tag{2.4}$$

provides us with three constants of motion, i.e.

$$\dot{J}_j = \{J_j, H\} = 0. \tag{2.5}$$

This condition is fulfilled provided

$$g^{ij}(\boldsymbol{x}) = G_1(x^2)\delta^{ij} + G_2(x^2) x^i x^j, \qquad \mathcal{U}(\boldsymbol{x}) = \mathcal{U}(x^2), \tag{2.6}$$

so that

$$H = \tfrac{1}{2}(G_1(x^2)\boldsymbol{p}^2 + G_3(x^2)(\boldsymbol{xp})^2) + \mathcal{U}(x^2). \tag{2.7}$$

If the covariant components of the metric tensor are written as

$$g_{ij}(\boldsymbol{x}) = d_1(x^2)\delta_{ij} + d_2(x^2) x_i x_j, \tag{2.8}$$

we have from (2.3)

$$G_1(x^2) d_1(x^2) = 1, \qquad (G_1(x^2) + x^2 G_2(x^2))(d_1(x^2) + x^2 d_2(x^2)) = 1. \tag{2.9}$$

The momenta and velocities are related by

$$p_j = \partial L / \partial \dot{x}^j = g_{jl} \dot{x}^l, \qquad \dot{x}^j = g^{jl} p_l. \tag{2.10}$$

The constants of the motion have the following form in terms of the velocities:

$$H = \tfrac{1}{2}(d_1(x^2)\dot{\boldsymbol{x}}^2 + d_2(x^2)(\boldsymbol{x}\dot{\boldsymbol{x}})^2) + \mathcal{U}(x^2), \tag{2.11}$$

$$J_j = d_1(x^2) \epsilon_{jkl} x^k \dot{x}^l, \tag{2.12}$$

$$J^2 = d_1^2(x^2)(x^2 \dot{\boldsymbol{x}}^2 - (\boldsymbol{x}\dot{\boldsymbol{x}})^2). \tag{2.13}$$

We shall also use spherical coordinates

$$x_1 = r \sin\theta \cos\phi, \qquad x_2 = r \sin\theta \sin\phi, \qquad x_3 = r\cos\theta, \tag{2.14}$$

in which (2.11)–(2.13) become

$$H = \tfrac{1}{2}(d_1(r^2) + r^2 d_2(r^2))\dot{r}^2 + \tfrac{1}{2} r^2 d_1(r^2)(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \mathcal{U}(r), \tag{2.15}$$

$$J_3 = r^2 d_1(r^2)\dot{\phi} \sin \theta, \tag{2.16}$$

$$J^2 = r^4 d_1^2(r^2)(\dot{\theta}^2 + \sin^2 \theta\, \dot{\phi}^2). \tag{2.17}$$

Combining (2.15) and (2.17) we can write

$$H = \tfrac{1}{2}(d_1(r^2) + r^2 d_2(r^2))\dot{r}^2 + J^2/2r^2 d_1(r^2) + \mathscr{U}(r^2). \tag{2.18}$$

3. The SO(4) or SO(3, 1) symmetric dynamical systems

Let us describe the dynamical system we discuss in our work and formulate the problem we solve in this section. We want to find all dynamical systems with Hamiltonians of the form (2.7) possessing three first integrals K_j, $j = 1, 2, 3$,

$$\dot{K}_j = \{K_j, H\} = 0, \tag{3.1}$$

in addition to the three components of the angular momentum (2.4), such that:

(i) The K_j constitute a vector and close together with J_j the Poisson bracket Lie algebra of SO(4) or SO(3, 1),

$$\{J_j, J_l\} = \epsilon_{jln} J_n, \tag{3.2}$$

$$\{J_j, K_l\} = \epsilon_{jln} K_n, \tag{3.3}$$

$$\{K_j, K_l\} = \eta \epsilon_{jln} J_n; \tag{3.4}$$

(ii) K_j is linear in momenta,

$$K_j = a(x^2)p_j + (\mathscr{C}_2(x^2)(xp) + \mathscr{C}_1(x^2))x_j. \tag{3.5}$$

Equation (3.1) imposes certain restrictions on the Hamiltonian H and Runge–Lenz vector K. They can be formulated as a system of ordinary differential equations for $a(x^2)$, $\mathscr{C}_1(x^2)$, $\mathscr{C}_2(x^2)$, $G_1(x^2)$, $G_2(x^2)$ and $\mathscr{U}(x^2)(f' \equiv df/dx^2)$:

$$(a + x^2\mathscr{C}_2)G_1' - \mathscr{C}_2 G_1 = 0, \tag{3.6}$$

$$(a + x^2\mathscr{C}_2)G_2' - (\mathscr{C}_2 + 2x^2\mathscr{C}_2')G_2 - 2\mathscr{C}_2' G_1 = 0, \tag{3.7}$$

$$2\mathscr{C}_1' G_1 + (2x^2\mathscr{C}_1' + \mathscr{C}_1)G_2 = 0, \tag{3.8}$$

$$(a + x^2\mathscr{C}_2)\mathscr{U}' = 0, \tag{3.9}$$

$$(2a' + \mathscr{C}_2)G_1 + (2x^2 a' - a)G_2 = 0, \tag{3.10}$$

$$G_1\mathscr{C}_1 = 0. \tag{3.11}$$

There is a further restriction on H and K imposed by the requirement (3.34) which allows one to express $\mathscr{C}_2(x^2)$ through $a(x^2)$:

$$\mathscr{C}_2(x^2) = (\eta + 2a(x^2)a'(x^2))/(a(x^2) - 2x^2 a'(x^2)). \tag{3.12}$$

We shall now find the solutions of the system (3.6)–(3.12). Equation (3.11) tells us that either

$$\mathscr{C}_1(x^2) = 0 \tag{3.13}$$

or

$$G_1(x^2) = 0. \tag{3.14}$$

We shall call the former the general case and the latter the degenerate case.

Let us discuss first the degenerate case. The system (3.6)–(3.11) reduces to

$$(a+x^2\mathscr{C}_2)G_2' - (\mathscr{C}_2 + 2x^2\mathscr{C}_2)G_2 = 0, \tag{3.15}$$

$$(2x^2\mathscr{C}_1' + \mathscr{C}_1)G_2 = 0, \tag{3.16}$$

$$(a+x^2\mathscr{C}_2)\mathscr{U}' = 0, \tag{3.17}$$

$$(2x^2a' - a)G_2 = 0. \tag{3.18}$$

$G_2(x^2) \equiv 0$ would yield a Hamiltonian independent of the momenta. So we assume $G_2(x^2) \not\equiv 0$, and then the solutions of equations (3.16) and (3.18) are

$$a(x^2) = a(1)\sqrt{x^2}, \qquad \mathscr{C}_1(x^2) = \mathscr{C}_1(1)/\sqrt{x^2}. \tag{3.19}$$

Note also that equation (3.12) becomes

$$0 \times \mathscr{C}_2(x^2) = \eta + 2a(x^2)a'(x^2). \tag{3.20}$$

This is inconsistent unless

$$a(1) = \sqrt{-\eta}, \tag{3.21}$$

in which case there is no restriction on $\mathscr{C}_2(x^2)$ imposed by (3.20).

It follows from (3.17) that either

$$\mathscr{U}(x^2) = \text{constant} \tag{3.22}$$

or

$$a(x^2) + x^2\mathscr{C}_2(x^2) = 0. \tag{3.23}$$

In the latter case one obtains

$$\mathscr{C}_2(x^2) = -a(x^2)/x^2 = -\sqrt{-3/x^2}, \tag{3.24}$$

while $\mathscr{U}(x^2)$ and $G_2(x^2)$ can be arbitrary functions. That is, the Hamiltonian and the Runge–Lenz vector are

$$H = \tfrac{1}{2}G_2(x^2)(xp)^2 + \mathscr{U}(x^2), \tag{3.25}$$

$$K_i = \sqrt{-\eta x^2}p_i + [-(xp)\sqrt{-\eta/x^2} + \mathscr{C}_1(1)/\sqrt{x^2}]x_i. \tag{3.26}$$

In the former case, $\mathscr{U} = \text{constant}$, $\mathscr{C}_2(x^2)$ remains arbitrary, while

$$G_2(x^2) = g(\sqrt{-\eta} + \sqrt{x^2}\mathscr{C}_2(x^2))^2, \qquad g = \text{constant}, \tag{3.27}$$

so that

$$H = \tfrac{1}{2}g(\sqrt{-\eta} + \sqrt{x^2}\mathscr{C}_2(x^2))^2(\boldsymbol{xp})^2, \tag{3.28}$$

$$K_i = \sqrt{-\eta x^2}p_i + [\mathscr{C}_2(x^2)(xp) + \mathscr{C}_1(1)/\sqrt{x^2}]x_i. \tag{3.29}$$

The expressions (3.26)–(3.29) are real only for $\eta = -1$.

The degenerate case is actually a realisation of a simpler underlying E(3) symmetry generated by the constants of motion J_i and

$$n_i = x_i/\sqrt{x^2}. \tag{3.30}$$

The Runge–Lenz vectors (3.26) and (3.29) are complex quantities built out of J and n:

$$\mathbf{K} = \sqrt{-\eta/x^2}(\mathbf{J} \times \boldsymbol{x}) + \mathscr{C}_1(1)\boldsymbol{n}, \tag{3.31}$$

$$\mathbf{K} = \sqrt{-\eta}(\mathbf{J} \times \boldsymbol{n}) + (\sqrt{2H/g} + \mathscr{C}_1(1))\boldsymbol{n}. \tag{3.32}$$

In the general case equations (3.8) and (3.11) are satisfied automatically and equation (3.9) implies again that either

$$\mathcal{U}(x^2) = \text{constant} \tag{3.33}$$

or

$$a(x^2) + x^2\mathcal{C}_2(x^2) = 0. \tag{3.34}$$

Assuming that (3.33) is the case, we find

$$G_1(x^2) = (1/g)(\eta x^2 + a^2(x^2)), \tag{3.35}$$

$$G_2(x^2) = G_1(x^2)(\eta + 4aa' - 4x^2a'^2)(a - 2x^2a')^{-2}. \tag{3.36}$$

The functions (3.12), (3.13), (3.33), (3.35) and (3.36) satisfy all the equations (3.6)–(3.11). The function $a(x^2)$ and the integration constant g (interaction constant) remain arbitrary. The Hamiltonian, the Lagrangian and the Runge–Lenz vector are

$$H = \frac{1}{2g}\left((\eta x^2 + a^2)p^2 + \frac{(\eta x^2 + a^2)(\eta + 4aa' - 4x^2a'^2)}{(a - 2x^2a')^2}(xp)^2\right) \tag{3.37}$$

$$L = \frac{1}{2}\left(\frac{g}{\eta x^2 + a^2}\dot{x}^2 - g\frac{\eta + 4aa' - 4x^2a'^2}{(\eta x^2 + a^2)^2}(x\dot{x})^2\right), \tag{3.38}$$

$$K_j = ap_j + \frac{\eta + 2aa'}{a - 2x^2a'}(xp)x_j. \tag{3.39}$$

One readily verifies that, for $\eta = 1$, this is a mechanical analogue of the Weinberg realisations of the chiral $SU(2) \times SU(2)$ symmetry in field theory (Weinberg 1968). We note that one of the two Casimir elements is zero,

$$\mathbf{JK} = 0, \tag{3.40}$$

and the Hamiltonian turns out to be proportional to the other,

$$H = (\eta/2g)(J^2 + \eta K^2). \tag{3.41}$$

In the subcase $a + x^2\mathcal{C}_2 = 0$ of the general case we obtain, inserting (3.12) in (3.34),

$$\eta x^2 + a^2(x^2) = 0 \tag{3.42}$$

or

$$a(x^2) = \sqrt{-\eta x^2}, \tag{3.43}$$

which is real only for $\eta = -1$. Now $\mathcal{U}(x^2)$ is arbitrary and

$$\mathcal{C}_2(x^2) = -\sqrt{-\eta/x^2}. \tag{3.44}$$

It follows from (3.6) that

$$G_1(x^2) = 0, \tag{3.45}$$

i.e. this subcase is actually contained in the degenerate case. It is (3.25) and (3.26) with $\mathcal{C}_1(1) = 0$.

4. The motion

In the degenerate case there are the constants of motion n_j fixing the direction of motion, i.e. the particle should move along a straight line. Choosing $\boldsymbol{n} = (1, 0, 0)$ so that

$\boldsymbol{x}=(x,0,0)$, $\boldsymbol{p}=(p,0,0)$, $\mathbf{K}=(K,0,0)$, we write (3.25) and (3.26) in the form

$$H=\tfrac{1}{2}x^2G_2(x^2)p^2+\mathscr{U}(x^2), \tag{4.1}$$

$$K=\mathscr{C}_1(1), \tag{4.2}$$

i.e. in the degenerate case the symmetry algebra, after reducing the problem to one dimension, disappears and the only constant of motion is the Hamiltonian.

In the general case equation (2.18) implies

$$\tfrac{1}{2}(d_1(r^2)+r^2d_2(r^2))\dot{r}^2+J/2r^2d_1(r^2)=E=\text{constant}, \tag{4.3}$$

and hence, taking into account (3.35) and (3.36),

$$g\int\frac{(a(r^2)-2r^2a'(r^2))\,\mathrm{d}r}{(a^2+\eta r^2)(2gE-\eta J^2-J^2a^2/r^2)^{1/2}}=t+c_1. \tag{4.4}$$

Solving this integral by the substitution

$$a(r^2)/r=2(2gE/J^2-\eta)z/(z^2+1) \tag{4.5}$$

one obtains

$$\left(\frac{1}{s_1}-s_1\right)\tan^{-1}\frac{z}{s_1}-\left(\frac{1}{s_2}-s_2\right)\tan^{-1}\frac{z}{s_2}=\frac{J(s_2^2-s_1^2)}{2g}(t-t_0) \tag{4.6}$$

provided $s_1^2\neq s_2^2$, and

$$(1-s_1^2)\frac{s_1z}{z^2+s_1^2}+(1+s_1^2)\tan^{-1}\frac{z}{s_1}=2s_1^3\frac{J}{2g}(t-t_0) \tag{4.7}$$

for $s_2^2=s_1^2$. Here

$$s_1=\frac{2gE}{J^2}-\eta+\left[1+\left(\frac{2gE}{J^2}-\eta\right)^2\right]^{1/2},\qquad s_2=\eta-\frac{2gE}{J^2}+\left[1+\left(\frac{2gE}{J^2}-\eta\right)^2\right]^{1/2} \tag{4.8}$$

Equations (4.5) and (4.6) or (4.7) define r as an implicit function of time t. It is remarkable that, due to the special form of s_1 and s_2, related by

$$s_1s_2=1, \tag{4.9}$$

equations (4.6) and (4.7) can be solved explicitly giving the results ($2gE-\eta J^2\geqslant 0$)

$$\frac{1}{r}a(r)=\sqrt{2gE-J^2}\sin\left(\sqrt{\frac{2E}{g}}(t-t_0)\right)\left(2gE\cos^2\sqrt{\frac{2E}{g}}(t-t_0)+J^2\sin^2\sqrt{\frac{2E}{g}}(t-t_0)\right)^{-1/2} \tag{4.10}$$

in the case of SO(4) symmetry ($\eta=1$) and

$$\frac{1}{r}a(r)=\sqrt{2gE+J^2}\,\mathrm{sh}\left(\sqrt{\frac{2E}{g}}(t-t_0)\right)\left(2gE\,\mathrm{ch}^2\sqrt{\frac{2E}{g}}(t-t_0)+J^2\,\mathrm{sh}^2\sqrt{\frac{2E}{g}}(t-t_0)\right)^{-1/2} \tag{4.11}$$

for the SO(3, 1) symmetry ($\eta=-1$). We note also that the trajectories prove to be closed for $\eta=1$. One could argue that this is a manifestation of the underlying symmetry. However, it has been demonstrated (Bacry *et al* 1966, Fradkin 1967, Mukunda 1967a, b) that any dynamical system with s degrees of freedom possesses an

SO($s+1$) dynamical symmetry group, so we would rather say that this is a consequence of the simple mode of realisation of the underlying dynamical symmetry.

It is convenient to replace the energy E by a frequency ω,

$$2E = \eta g\omega^2, \tag{4.12}$$

in equations (4.10) and (4.11) and in what follows. Note that here the frequency is a dynamical variable and not merely an external parameter. The two expressions (4.10) and (4.11) can be combined in a single expression

$$a(r)/r = \sqrt{\eta(g^2\omega^2 - J^2)}\sin\omega(t-t_0)[g^2\omega^2\cos^2\omega(t-t_0) + J^2\sin^2\omega(t-t_0)]^{-1/2}, \tag{4.13}$$

with $\omega^2 = 2\eta E/g$ ($2gE \geqslant J^2$ for SO(4) and $2gE \geqslant -J^2$ for SO(3, 1)).

In order to find the trajectory, we solve the differential equations (2.16) and (2.17). Choosing the coordinate system so that $J_3 = 0$, we obtain from (2.16)

$$\dot\phi = 0, \qquad \phi = \phi_0 = \text{constant}, \tag{4.14}$$

and (2.17) takes the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{J}{r^2}G_1(r^2) = \frac{J}{g}\left(\eta + \frac{a^2(r)}{r^2}\right). \tag{4.15}$$

Combining (4.15) with (4.4) we have

$$J\int \frac{a(r^2) - 2r^2\,\mathrm{d}a/\mathrm{d}r^2}{r^2(2gE - \eta J^2 - J^2a^2/r^2)^{1/2}}\,\mathrm{d}r = \theta - \theta_0, \tag{4.16}$$

which yields

$$a(r)/r = \sqrt{\eta(g^2\omega^2 - J^2)}\sin(\theta - \theta_0)/J. \tag{4.17}$$

Another way of obtaining the same result is to observe that on account of (3.39) and (3.41) the equality

$$K_ix_i = |\mathbf{K}|r\cos(\theta - \theta_0) \tag{4.18}$$

takes the form

$$[(\eta r^2 + a^2)/(a - 2r^2a')](xp) = \sqrt{2gH - \eta J^2}\,r\cos(\theta - \theta_0), \tag{4.19}$$

and eliminating the momenta from here and

$$G_1(r^2)p^2 + G_2(r^2)(xp)^2 = 2H, \qquad r^2p^2 - (xp)^2 = J^2, \tag{4.20}$$

we obtain (2.17).

It follows from (4.13) and (4.17) that

$$\sin(\theta - \theta_0) = J\sin\omega(t-t_0)[g^2\omega^2\cos^2\omega(t-t_0) + J^2\sin^2\omega(t-t_0)]^{-1/2} \tag{4.21}$$

or

$$\sin\theta = [g\omega\sin\theta_0\cos\omega(t-t_0) - J\cos\theta_0\sin\omega(t-t_0)]$$
$$\times[g^2\omega^2\cos^2\omega(t-t_0) + J^2\sin^2\omega(t-t_0)]^{-1/2}. \tag{4.22}$$

Equations (4.13), (4.14) and (4.22) describe completely the motion of the particle on the (closed) trajectory (4.17). The motion and the trajectory depend on the choice of arbitrary function $a(r)$.

The functions

$$a(r)=\sqrt{g-r^2}, \tag{4.23}$$

$$a(r)=\sqrt{g}(1-r^2/4g), \tag{4.24}$$

$$a(r)=r\cot(r/\sqrt{g}) \tag{4.25}$$

are used most frequently in the literature on SU(2)×SU(2) chiral field models. The choice (4.23),

$$a(r)=\sqrt{g-\eta r^2}, \tag{4.26}$$

is convenient for the following discussion. We have in this case

$$\begin{aligned} r&=[g\cos^2\omega(t-t_0)+(J^2/g\omega^2)\sin^2\omega(t-t_0)]^{1/2},\\ r\sin\theta&=\sqrt{g}[\sin\theta_0\cos\omega(t-t_0)-(J/g\omega)\cos\theta_0\sin\omega(t-t_0)],\\ \phi&=\phi_0, \end{aligned} \tag{4.27}$$

or in Cartesian coordinates

$$\begin{aligned} x_1&=\sqrt{g}[\sin\theta_0\cos\phi_0\cos\omega(t-t_0)-(J/g\omega)\cos\theta_0\cos\phi_0\sin\omega(t-t_0)],\\ x_2&=\sqrt{g}[\sin\theta_0\cos\phi_0\cos\omega(t-t_0)-(J/g\omega)\cos\theta_0\sin\phi_0\sin\omega(t-t_0)],\\ x_3&=\sqrt{g}[\cos\theta_0\cos\omega(t-t_0)+(J/g\omega)\sin\theta_0\sin\omega(t-t_0)]. \end{aligned} \tag{4.28}$$

Equation (4.28) may be written in the more compact form

$$\boldsymbol{x}=\sqrt{g}[\boldsymbol{n}\cos\omega(t-t_0)+(J/g\omega)\boldsymbol{l}\sin\omega(t-t_0)], \tag{4.29}$$

where $\boldsymbol{n}$ and $\boldsymbol{l}$ are two unit orthogonal vectors,

$$\boldsymbol{n}^2=1=\boldsymbol{l}^2, \qquad \boldsymbol{nl}=0, \tag{4.30}$$

with components

$$\begin{aligned} n_1&=\sin\theta_0\cos\phi_0, & l_1&=\sin(\theta_0-\pi/2)\cos\phi_0,\\ n_2&=\sin\theta_0\sin\phi_0, & l_2&=\sin(\theta_0-\pi/2)\sin\phi_0,\\ n_3&=\cos\theta_0, & l_3&=\cos(\theta_0-\pi/2). \end{aligned} \tag{4.31}$$

5. Transformation properties and geometric interpretation

We have demonstrated in the preceding section that in the special case (4.26) the solution (4.29) is a superposition of two simple harmonic oscillations. Knowing the solution x for one choice of $a(x^2)$, we can find the solution $\tilde{x}$ for any other choice of $\tilde{a}(\tilde{x}^2)$ by

$$\tilde{x}_i=x_i\Phi(x^2). \tag{5.1}$$

The function $\Phi(x^2)$ satisfies the equation (Weinberg 1968)

$$\tilde{a}(x^2\Phi^2(x^2))=a(x^2)\Phi(x^2). \tag{5.2}$$

One should reproduce in this way the solution (4.13), (4.14), (4.22).

All these general case solutions as well as their particular forms (4.28) or (4.29) depend only on five integration constants: ω, J, t_0, θ_0, and ϕ_0 (rather than on six). Our aim in the present section is to enlarge this family of solutions by increasing the number of independent integration constants to its maximum value of six.

Before doing so, let us summarise the transformation properties of x_j under the transformations of the symmetry group. The infinitesimal transformations in configuration space are given by

$$\tilde{x}_j = x_j + \alpha_l\{J_l, x_j\} + \beta_l\{K_l, x_j\}, \tag{5.3}$$

with

$$\{J_l, x_j\} = \epsilon_{ljn}x_n, \tag{5.4}$$

$$\{K_l, x_j\} = -a(x^2)\delta_{lj} - [(\eta + 2a(x^2)a'(x^2))/(a(x^2) - 2x^2a'(x^2))]x_lx_j. \tag{5.5}$$

Any element X of the Lie algebra of the symmetry group generates a one-parameter subgroup of transformations which act on an arbitrary dynamical variable F according to

$$F \to \tilde{F} = e^{\alpha X} * F \equiv F + \frac{\alpha}{1!}\{X, F\} + \frac{\alpha^2}{2!}\{X, \{X, F\}\} + \dots. \tag{5.6}$$

The transformations generated by the J's are just the space rotations

$$\tilde{x} = e^{\alpha_l J_l} * x = R(\alpha_1, \alpha_2, \alpha_3)x. \tag{5.7}$$

Here $R(\alpha_1, \alpha_2, \alpha_3)$ is the matrix of the space rotations, and the α's are the rotation angles about the three axes. Applying this linear transformation to the solutions (4.29) we obtain

$$\tilde{\boldsymbol{x}} = R\boldsymbol{x} = \sqrt{g}[\tilde{\boldsymbol{n}} \cos \omega(t - t_0) + (J/g\omega)\tilde{\boldsymbol{l}} \sin \omega(t - t_0)], \tag{5.8}$$

$$\tilde{\boldsymbol{n}} = R\boldsymbol{n}, \qquad \tilde{\boldsymbol{l}} = R\boldsymbol{l}, \tag{5.9}$$

and this is the desired six-parameter family of solutions. Indeed, due to the fact that the rotations preserve the orthonormality relations, (4.30) goes into

$$\tilde{\boldsymbol{n}}^2 = 1 = \tilde{\boldsymbol{l}}^2, \qquad \tilde{\boldsymbol{n}}\tilde{\boldsymbol{l}} = 0, \tag{5.10}$$

and the two vectors $\tilde{\boldsymbol{n}}, \tilde{\boldsymbol{l}}$ constrained by (5.10) bring into equation (5.8) $6 - 3 = 3$ independent parameters in addition to J, ω, t_0, so that the total number of independent parameters is six. One can write (5.8) or (4.29) in the form

$$\boldsymbol{x} = \sqrt{g}(\mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t), \tag{5.11}$$

with

$$\mathbf{A} = \boldsymbol{n} \cos \omega t_0 - (J/g\omega)\boldsymbol{l} \sin \omega t_0, \qquad \mathbf{B} = \boldsymbol{n} \sin \omega t_0 + (J/g\omega)\boldsymbol{l} \cos \omega t_0 \tag{5.12}$$

obeying the only restriction

$$\mathbf{A}^2 + \mathbf{B}^2 - (\mathbf{A} \times \mathbf{B})^2 = 1. \tag{5.13}$$

The equations of motion in Newtonian form,

$$\ddot{x}^j + \Gamma^j_{mn}\dot{x}^m\dot{x}^n = 0, \tag{5.14}$$

with the Christoffel symbol

$$\Gamma^j_{mn} = \tfrac{1}{2}g^{js}(\partial g_{sn}/\partial x^m + \partial g_{ms}/\partial x^n - \partial g_{mn}/\partial x^s) \tag{5.15}$$

given by

$$\Gamma^{i}_{mn} = -\frac{\eta + 2aa'}{\eta x^2 + a^2}(x_m\delta^i_n + x_n\delta^i_m) - \frac{2a'}{a - 2x^2 a'}x^i\delta_{mn} - \frac{4a''}{a - 2x^2 a'}x^i x_m x_n, \tag{5.16}$$

have the geometrical meaning of the geodesics in a curved space of constant curvature. For the parameterisation (4.26), the Christoffel symbol is

$$\Gamma^{i}_{mn} = (1/g)[\eta x^i\delta_{mn} + x^i x_m x_n/(g - \eta x^2)], \tag{5.17}$$

and the equations of motion (5.14) take the form

$$g(g - \eta x^2)\ddot{x}_i + [\eta(g - \eta x^2)\dot{x}^2 + (x\dot{x})^2]x_i = 0. \tag{5.18}$$

We already know that they have the solutions (5.11) with $\boldsymbol{A}$ and $\boldsymbol{B}$ arbitrary vectors restricted by (5.13). We can verify directly that this is really a solution of (5.18). To do this we note that (5.11) always implies

$$\ddot{\boldsymbol{x}} = -\omega^2\boldsymbol{x} \tag{5.19}$$

and *vice versa*, so that on the set of functions (5.11) equation (5.18) is equivalent to

$$\eta g(\dot{x}^2 + \omega^2 x^2) - [\dot{x}^2 x^2 - (x\dot{x})^2] = \omega^2 g^2. \tag{5.20}$$

Inserting here (5.11) we obtain

$$\eta(A^2 + B^2) - (A \times B)^2 = 1, \tag{5.21}$$

coinciding with (5.13) in the case $\eta = 1$. In the case $\eta = -1$, $\boldsymbol{B}$ as well as ω should be imaginary.

We end the section by studying the action of the transformations generated by K on the solution (5.11), (5.13) for $\eta = 1$. It follows from the relations

$$\{\boldsymbol{\beta}\mathbf{K}, \sqrt{g - x^2}\} = \boldsymbol{\beta x}, \qquad \{\boldsymbol{\beta}\mathbf{K}, \boldsymbol{\beta x}\} = -\beta^2\sqrt{g - x^2} \tag{5.22}$$

implied by (5.5) that (cf the definition (5.6))

$$\tilde{x}_i \equiv e^{(\beta K)} * x_i = x_i + \left(\frac{\cos\beta - 1}{\beta^2}(\beta x) - \frac{\sin\beta}{\beta}\sqrt{g - x^2}\right)\beta_i. \tag{5.23}$$

Here $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ are the parameters of the transformation and $\beta = |\boldsymbol{\beta}|$. Inserting in the right-hand side of (5.23) the solution (5.11), (5.13) we obtain

$$\tilde{\boldsymbol{x}} = \sqrt{g}(\tilde{\mathbf{A}}\cos\omega t + \tilde{\mathbf{B}}\sin\omega t), \tag{5.24}$$

where

$$\begin{aligned} \tilde{A}_i &= A_i + (\boldsymbol{\beta}\mathbf{A})\frac{\cos\beta - 1}{\beta^2}\beta_i - \sqrt{1 - A^2}\,\frac{\sin\beta}{\beta}\beta_i \equiv Q_i(\boldsymbol{A}; \boldsymbol{\beta}), \\ \tilde{B}_i &= B_i + (\boldsymbol{\beta}\mathbf{B})\frac{\cos\beta - 1}{\beta^2}\beta_i \pm \sqrt{1 - B^2}\,\frac{\sin\beta}{\beta}\beta_i = Q_i(\boldsymbol{B}; \mp\boldsymbol{\beta}). \end{aligned} \tag{5.25}$$

The functions Q_i define the non-linear action of the transformations generated by $\mathbf{K}$ in the three-dimensional real space $\mathbb{R}^3$ of the vectors $\mathbf{A}$, $\mathbf{B}$ or in the six-dimensional space $\mathbb{R}^6$ of the vectors (A, B). They satisfy the functional equations

$$Q[Q(\boldsymbol{A}; \boldsymbol{\beta}); \gamma] = Q[\boldsymbol{A}; \boldsymbol{\kappa}(\boldsymbol{\beta}, \boldsymbol{\gamma})], \tag{5.26}$$

equivalent to

$$e^{\gamma \mathbf{K}} * (e^{\beta \mathbf{K}} * x) = e^{\kappa(\beta,\gamma)\mathbf{K}} * x. \tag{5.27}$$

It is not difficult to determine the composition function

$$\boldsymbol{\alpha} = \boldsymbol{\kappa}(\boldsymbol{\beta}, \boldsymbol{\gamma}) \tag{5.28}$$

directly from (5.26). The result is

$$\pm\cos \alpha = \cos \beta \cos \gamma - \frac{(\beta\gamma)}{\beta\gamma} \sin \beta \sin \gamma, \tag{5.29}$$

$$\frac{\alpha_i}{\alpha} \sin \alpha = \frac{\beta_i}{\beta} \sin \beta + \frac{\gamma_i}{\gamma}\left(\sin \gamma \cos \beta + \frac{(\beta\gamma)}{\beta\gamma} \sin \beta (\cos \gamma - 1)\right). \tag{5.30}$$

The transformation

$$(\mathbf{A}, \mathbf{B}) \overset{Q}{\rightarrow} (\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \tag{5.31}$$

defined by (5.25) keeps intact the constraint (5.13) on the integration constants:

$$\tilde{\mathbf{A}}^2 + \tilde{\mathbf{B}}^2 - (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})^2 = 1, \tag{5.32}$$

which is another way of saying that the solution (5.11) goes into a solution (5.24) under the transformation (5.23) generated by K.

Let us make a recapitulation. There are six independent constants of motion in our problem. One of them is the frequency ω (the energy E, cf (4.12)). The manifold of the other five is a fourth-degree surface

$$S\colon (AB)^2 - A^2B^2 + \eta A^2 + \eta B^2 - 1 = 0 \tag{5.33}$$

in $\mathbb{R}^6$. The frequency ω remains unchanged under the transformations of the symmetry group SO(4) or SO(3, 1), while the surface (5.33) is mapped onto itself.

We can gain a further insight into the picture of the motion by recovering the fourth dimension of the vectors x, A and B. Namely, define the four-dimensional vectors $q = (q_0, q_1, q_2, q_3)$, $a = (a_0, a_1, a_2, a_3)$ and $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3)$ by

$$\begin{aligned} &q_j = x_j/\sqrt{g}, \qquad a_j = A_j, \qquad \mathscr{C}_j = B_j, \qquad j = 1, 2, 3, \\ &a^2 \equiv a_0^2 + \eta \mathbf{a}^2 = 1, \qquad \mathscr{C}^2 \equiv \mathscr{C}_0^2 + \eta \boldsymbol{\mathscr{C}}^2 = \eta. \end{aligned} \tag{5.34}$$

Then the surface S can be represented as a manifold imbedded in $\mathbb{R}^8$ defined by

$$a^2 = 1, \qquad \mathscr{C}^2 = \eta, \qquad a\mathscr{C} = 0. \tag{5.35}$$

The motion law (5.11),

$$q_j = a_j \cos \omega t + \mathscr{C}_j \sin \omega t, \tag{5.36}$$

can now be completed by

$$q_0 = a_0 \cos \omega t + \mathscr{C}_0 \sin \omega t \tag{5.37}$$

to a motion on the unit sphere (or hyperboloid for $\eta = -1$) in

$$q^2 \equiv q_0^2 + \eta \mathbf{q}^2 = 1. \tag{5.38}$$

The last equality follows from (5.35). This allows us to conceive the motion (5.10) as a three-dimensional projection of the four-dimensional motion (5.36), (5.37) on the sphere (5.38).

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